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Orthogonal Polynomials and Exact Correlation Functions for Two Cut Random Matrix Models

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Abstract

Exact eigenvalue correlation functions are computed for large N hermitian one-matrix models with eigenvalues distributed in two symmetric cuts. An asymptotic form for orthogonal polynomials for arbitrary polynomial potentials that support a Z_2 symmetric distribution is obtained. This results in an exact explicit expression for the kernel at large N which determines all eigenvalue correlators. The oscillating and smooth parts of the two point correlator are extracted and the universality of local fine grained and smoothed global correlators is established.

1 Introduction

Eigenvalue correlators in large random matrix models have been of interest recently (see [1, 2, 3, 4, 5, 6, 7, 8, 9] and references therein). The recent interest is in part inspired by the burst of activity in the field of quantum chaos and mesoscopic systems.

Correlators in matrix models have been calculated in various limits by different methods. At short separations of the order of the eigenvalue spacing the correlators show oscillatory behaviour. Correlators that reproduce this behaviour are referred to as "fine grained". If these oscillations are averaged over, one gets the so called "smoothed" correlators. For some applications, one needs the correlation functions over the entire eigenvalue distribution, near the centre as well as the edges (global correlators) while for others only in a small region near the centre or the edge of the eigenvalue distribution (local correlators). The complete information is contained in the global fine grained correlator, from which all others can be obtained as special limiting cases.

The global fine grained correlator has been computed for the hermitian one matrix model [1] and the hermitian two matrix model [10] using the method of orthogonal polynomials. These studies, however, consider only those cases where the eigenvalue distribution lies in a single cut. The present paper is concerned with correlation functions for hermitian matrix models in which eigenvalues lie in two cuts or bands.

Random matrix models with eigenvalues that have support in more than one cut have been studied in the context of 2-d quantum gravity and symmetry breaking solutions ([11] and references therein), two-dimensional QCD[12, 13] and statistical physics [14]. They may also be applicable in studies of certain condensed matter systems with eigenvalue distributions with gaps. In the present paper the global fine grained correlators for matrix models with eigenvalue support in two symmetric cuts are obtained. These expressions therefore contain detailed information about how the correlators vary over distances of the eigenvalue spacing, from which the smoothed correlators can be extracted by a suitable averaging procedure. They are also valid over the entire eigenvalue distribution, near the centre as well as the edge of the cuts.

In order to compute the global fine grained correlators, an asymptotic form for orthogonal polynomials is derived, which is valid for arbitrary potentials that support a Z_2 symmetric eigenvalue distribution. This form allows a determination of an explicit expression for the eigenvalue kernel

through the Christoffel-Darboux formula suitably modified for the two cut situation. The kernel then determines all the global fine grained n-point correlation functions of the eigenvalue density.

The two point function is explicitly discussed in various limits. The kernel as well as the global fine grained correlator in general depend upon the coupling constants in the matrix model potential. However, it is shown that in two limits the two point function is universal, i.e., independent of the form of the potential. These two limits are (i) fine grained local correlators away from the edge of the cuts, where the Dyson form is reproduced, and (ii) global smoothed correlators.

This generalizes the work of [1] to the two cut matrix models. When the coupling constants of the matrix model change such that the eigenvalue support moves from two cuts to one cut, the single-cut results are obtained. Thus the present work can be used to study this phase transition in the coupling constant space of matrix models at the level of fine grained correlators, generalizing earlier studies at the level of smoothed correlators [15, 16].

The paper is organized as follows. In the second section after establishing notations and conventions, the asymptotic form for the orthogonal polynomials for two symmetric cuts is presented as an ansatz. In the third and fourth sections the recurrence relation and the orthogonality condition are used to determine the unknown functions in the ansatz. The fifth section derives the Christoffel-Darboux formula for the kernel $K(\mu, \nu)$ in a form suitable for two cuts and an explicit expression for kernel is presented. The sixth section discusses the two point correlation function in its full fine grained global form and establishes its universality in various limits (fine grained local, smoothed global). The seventh section contains concluding remarks.

2 Notation, Conventions and Asymptotic Ansatz

The partition function is defined by

$$Z \equiv \int dM \, \exp(-N \, \mathrm{Tr} V(M)) \, , \tag{2.1}$$

where M is an $N \times N$ hermitian matrix and $V(M) \equiv \sum_{n=1}^{\infty} \frac{g_n}{n} M^n$ is a real polynomial in M (the same notation as ref. [11] is used). $P_n(\lambda)$ is a set of polynomials that are orthogonal with respect to the measure defined by the potential V, $\int_{-\infty}^{\infty} d\lambda P_n(\lambda) P_m(\lambda) exp(-N \, \mathrm{Tr} V(\lambda)) = h_n \delta_{nm}$, and are

normalized such that $P_n(\lambda) = \lambda^n + c_1\lambda^{n-1} + ...$, $P_0(\lambda) = 1$. These properties determine $P_n(\lambda)$ uniquely for every V. The orthogonal polynomials satisfy the recursion relation $\lambda P_n(\lambda) = P_{n+1}(\lambda) + S_n P_n(\lambda) + R_n P_{n-1}(\lambda)$ where the recursion coefficients R_n, S_n are V dependent. $R_n = h_n/h_{n-1}$ and for Z_2-symmetric potentials $(V(-\lambda) = V(\lambda))$ $S_n = 0$. It is convenient to define an orthonormal set of polynomials $\psi_n(\lambda) = \frac{P_n(\lambda)}{\sqrt{h_n}} exp(-\frac{N}{2}\ \mathrm{Tr}V(\lambda))$. Then

$$\int d\lambda \psi_n(\lambda)\psi_m(\lambda) = \delta_{nm}, \tag{2.2}$$

and

$$\lambda\psi_n(\lambda) = \sqrt{R_{n+1}}\psi_{n+1}(\lambda) + \sqrt{R_n}\psi_{n-1}(\lambda). \tag{2.3}$$

The kernel $K(\mu,\nu)$ is defined as

$$K(\mu,\nu) = \frac{1}{N}\sum_{i=0}^{N-1}\psi_i(\mu)\psi_i(\nu). \tag{2.4}$$

The Christoffel-Darboux formula expresses this in terms of just ψ_n with n close to N:

$$K(\mu,\nu) = \frac{\sqrt{R_N}}{N}\frac{[\psi_N(\mu)\psi_{N-1}(\nu) - \psi_{N-1}(\mu)\psi_N(\nu)]}{(\mu-\nu)}. \tag{2.5}$$

The kernel determines all eigenvalue correlators. Defining $\hat\rho(\mu) = \frac{1}{N}Tr\delta(\mu - M) = \frac{1}{N}\sum_{i=1}^{N}\delta(\mu-\lambda_i)$, where λ_i are the eigenvalues of M, it follows that

$$\begin{aligned}
\rho(\mu) &\equiv \langle\hat\rho(\mu)\rangle = K(\mu,\mu) \\
\rho_c(\mu,\nu) &\equiv \langle\hat\rho(\mu)\hat\rho(\nu)\rangle - \langle\hat\rho(\mu)\rangle\langle\hat\rho(\nu)\rangle = -[K(\mu,\nu)]^2 \\
\rho_c(\lambda,\mu,\nu) &\equiv \langle\hat\rho(\lambda)\hat\rho(\mu)\hat\rho(\nu)\rangle_c = 2K(\lambda,\mu)K(\mu,\nu)K(\nu,\lambda)
\end{aligned} \tag{2.6}$$

where the subscript c stands for connected correlator and the expectation value of any function $f(M)$ is given by $\langle f\rangle = \frac{1}{Z}\int dMe^{-NTrV(M)}f(M)$.

When the potential is Z_2 symmetric and has two wells the eigenvalue density $\rho(\lambda)$ can have support in two cuts located symmetrically about the origin, $\lambda \in (a,b)\cup(-b,-a)$, $a \geq 0, b > a$. One is then interested in the kernel and correlation functions where μ,ν, etc. lie in the cuts. The Christoffel-Darboux formula implies that we need the orthogonal polynomials for n close to N.

For λ lying in the two cuts and for $N-n \sim O(1)$, N large, we make the ansatz that the orthogonal polynomials can be approximated by

$$\psi_n(\lambda) = \frac{1}{\sqrt{f}}[\cos(N\zeta - (N-n)\phi + \chi + (-1)^n\eta) + O(\frac{1}{N})], \tag{2.7}$$

where f, ζ, ϕ, χ and η are functions of λ, and that ψ_n is damped out outside the cuts. We then show that

$$f(\lambda) = \frac{\pi}{2\lambda}\frac{(b^2-a^2)}{2}\sin 2\phi(\lambda) \tag{2.8}$$

from the orthogonality condition satisfied by the orthogonal polynomials and is derived in section 4. $\zeta(\lambda)$ is fixed to be

$$\zeta'(\lambda) = -\pi\rho(\lambda) \tag{2.9}$$

from the relation $K(\lambda,\lambda) = \rho(\lambda)$ and is derived in section 5. $\phi(\lambda)$ and $\eta(\lambda)$ are determined from the recurrence relation satisfied by the orthogonal polynomials (section 3)

$$\cos 2\phi(\lambda) = \frac{\lambda^2 - \frac{(a^2+b^2)}{2}}{\frac{(b^2-a^2)}{2}}, \tag{2.10}$$

$$\begin{aligned} \cos 2\eta(\lambda) &= b\frac{\cos\phi(\lambda)}{\lambda}, \\ \sin 2\eta(\lambda) &= a\frac{\sin\phi(\lambda)}{\lambda}. \end{aligned} \tag{2.11}$$

This leaves the function $\chi(\lambda)$ undetermined and we conjecture that it has the form $\frac{1}{2}\phi(\lambda) - \frac{\pi}{4}$ for all potentials as in the single cut case ref. ([1, 17]).

Example for $g_2 < 0, g_4 > 0$, all other $g_i = 0$, and in the two cut phase $g_2^2 > 4g_4$, the end points are given by $a^2 = -2\sqrt{\frac{1}{g_4}} - \frac{g_2}{g_4}$ and $b^2 = 2\sqrt{\frac{1}{g_4}} - \frac{g_2}{g_4}$. The density of eigenvalues is $\rho(\lambda) = \frac{g_4}{2\pi}\lambda\sqrt{(\lambda^2-a^2)(b^2-\lambda^2)}$ for $\lambda \in (-b,-a) \cup (a,b)$ and zero elsewhere. Hence $\zeta(\lambda) = \phi(\lambda) - \frac{\sin 4\phi(\lambda)}{4}$. This determines the asymptotic form of $\psi_n(\lambda)$ completely.

Note that f, ϕ, η and χ are universal functions, i.e, their functional form is independent of the potential V, the only dependence on V enters through the endpoints of the cuts a and b. ζ is non-universal since the eigenvalue density depends in general on the detailed form of V.

3 Recurrence Relation

We will now use Eq. (2.3) to determine ϕ and η. Multiplying Eq. (2.3) by λ and using Eq. (2.3) once again we get

$$\begin{aligned}\lambda^2\psi_n(\lambda) &= \sqrt{R_{n+1}}\sqrt{R_{n+2}}\psi_{n+2}(\lambda) + R_{n+1}\psi_n(\lambda) + R_n\psi_n(\lambda) \\ &+ \sqrt{R_n}\sqrt{R_{n-1}}\psi_{n-2}(\lambda). \end{aligned} \tag{3.1}$$

When V has two symmetric wells, it is known that $R_n = A_n$ for n even and $R_n = B_n$ for n odd, where in the large N limit A_n and B_n are approximated by two continuous functions $A(x)$, $B(x)$ with $x = \frac{n}{N}$ [18, 16]. Thus, for n even

$$\begin{aligned}\lambda^2\psi_n(\lambda) &= \sqrt{B_{n+1}}\sqrt{A_{n+2}}\psi_{n+2}(\lambda) + B_{n+1}\psi_n(\lambda) \\ &+ A_n\psi_n(\lambda) + \sqrt{A_n}\sqrt{B_{n-1}}\psi_{n-2}(\lambda). \end{aligned} \tag{3.2}$$

On using the asymptotic ansatz for even n i.e., substituting Eq. (2.7) in Eq. (3.2), and replacing $A_n = A(x = \frac{n}{N}) = A(x = 1 - \frac{N-n}{N}) = A(1) + O(\frac{1}{N}) \approx A(1) \equiv A$ (and similarly $B_n \to B(1) \equiv B$) we get

$$\frac{\lambda^2 - (A+B)}{2\sqrt{AB}} = \cos 2\phi(\lambda). \tag{3.3}$$

It is known that for Z_2 symmetric two cut models the end points a and b are related to A and B by [16]

$$\begin{aligned} A + B &= \frac{a^2+b^2}{2}, \\ 2\sqrt{AB} &= \frac{b^2-a^2}{2}. \end{aligned} \tag{3.4}$$

This yields Eq. (2.10).

Next consider the recurrence relations (n even)

$$\begin{aligned} \lambda\psi_{n+1}(\lambda) &= \sqrt{R_{n+2}}\psi_{n+2}(\lambda) + \sqrt{R_{n+1}}\psi_n(\lambda) \\ \lambda\psi_{n-1}(\lambda) &= \sqrt{R_n}\psi_n(\lambda) + \sqrt{R_{n-1}}\psi_{n-2}(\lambda). \end{aligned} \tag{3.5}$$

On substituting Eq. (2.7) for ψ_n, $\psi_{n\pm1}$, $\psi_{n\pm2}$ in Eq. (3.5) it is easy to see that

$$\begin{aligned} \sin 2\eta(\lambda) &= (-\sqrt{A} + \sqrt{B})\frac{\sin\phi(\lambda)}{\lambda} \\ &= a\frac{\sin\phi(\lambda)}{\lambda} \end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\cos 2\eta(\lambda) &= (\sqrt{A}+\sqrt{B})\frac{\cos\phi(\lambda)}{\lambda} \\ &= b\frac{\cos\phi(\lambda)}{\lambda}. \end{aligned} \tag{3.7}$$

This determines Eqs. (2.10) and (2.11).

4 Orthogonality

Let us check for orthonormality, Eq. (2.2), for all n, m. The ansatz assumes that for large N $\psi_n(\lambda)$ is damped out sufficiently fast beyond the end of cuts so that integration can be restricted to the cuts

$$(\int_{-b}^{-a} + \int_a^b) d\lambda \psi_n(\lambda)\psi_m(\lambda) = \delta_{nm}. \tag{4.1}$$

The integrand is

$$\begin{aligned}\psi_n\psi_m &= \frac{1}{f}\cos\alpha\cos\beta \\ &= \frac{1}{2f}[\cos(\alpha+\beta)+\cos(\alpha-\beta)] \end{aligned} \tag{4.2}$$

where $(\alpha+\beta) = (2N\zeta - (2N-n-m)\phi + 2\chi + (-1)^n\eta)$ and $(\alpha-\beta) = ((n-m)\phi + ((-1)^n - (-1)^m)\eta)$. The $\cos(\alpha+\beta)$ term is zero upon integration because $\cos(2N\zeta - (2N-n-m)\phi + 2\chi + (-1)^n\eta)$ oscillates rapidly and averages to zero. Thus

$$(\psi_n, \psi_m) = (\int_{-b}^{-a} + \int_a^b)\frac{d\lambda}{2f}\cos(\alpha-\beta) \tag{4.3}$$

Note that $(-1)^n - (-1)^m = 0$ if $n-m$ is even and $(-1)^n - (-1)^m = 2(-1)^n$ if $n-m$ is odd. For $n-m$ even

$$(\psi_n, \psi_m) = (\int_{-b}^{-a} + \int_a^b)\frac{d\lambda}{2f}\cos(n-m)\phi. \tag{4.4}$$

In particular for $n-m=0$

$$(\psi_n, \psi_n) = (\int_{-b}^{-a} + \int_a^b)\frac{d\lambda}{2f}. \tag{4.5}$$

We will show that if

$$\frac{d\lambda}{d\phi} = cf(\lambda), \tag{4.6}$$

then orthogonality follows. On taking the derivative of Eq. (2.10) one gets

$$\frac{d\lambda}{d\phi} = \frac{-1}{\lambda}\frac{(b^2-a^2)}{2}\sin 2\phi \tag{4.7}$$

Thus

$$f(\lambda) = \frac{-1}{c}\frac{(b^2-a^2)}{2}\frac{\sin 2\phi(\lambda)}{\lambda}. \tag{4.8}$$

Note from Eq. (4.5) that for $(\psi_n, \psi_n) = 1$, $f(\lambda)$ must be an even function of λ. This implies that $\sin 2\phi$ must be an odd function of λ. Since $\cos 2\phi = -1$ at $\lambda = \pm a$, and 1 at $\lambda = \pm b$, this is achieved by taking $2\phi = 2p\pi, (2p-1)\pi, (2p-3)\pi, (2p-4)\pi$ at $\lambda = -b, -a, a, b$ respectively; with p any integer. With this choice $\sin 2\phi$ is odd because 2ϕ is in the first two quadrants for λ in (a, b) and the third and fourth quadrants for λ in $(-b, -a)$. In particular this choice also implies that $\cos\phi$ is an even function of λ and $\sin\phi$ is an odd function. Substituting Eq. (4.8) into Eq. (4.5) implies that $c = \frac{-2}{\pi}$. Using $c = \frac{-2}{\pi}$ in Eq. (4.8) gives Eq. (2.8).

Now consider $n - m \neq 0$ even. Using Eq. (4.6) in Eq. (4.4), we find

$$\begin{aligned} (\psi_n, \psi_m) &= c(\int_{\phi(-b)}^{\phi(-a)} + \int_{\phi(a)}^{\phi(b)})d\phi\cos(n-m)\phi \\ &= 0. \end{aligned} \tag{4.9}$$

Though the integrand is an even function of λ the full integral vanishes as each integral separately vanishes.

Now consider $n - m$ odd

$$(\psi_n, \psi_m) = (\int_{-b}^{-a} + \int_a^b)\frac{d\lambda}{2f}\cos(\alpha - \beta) \tag{4.10}$$

where $\alpha - \beta = (n-m)\phi + 2(-1)^n\eta$ for $n - m$ odd. Expanding $\cos(\alpha - \beta)$ we get

$$(\psi_n, \psi_m) = (\int_{-b}^{-a} + \int_a^b)\frac{d\lambda}{2f}[\cos(n-m)\phi\cos 2\eta - (-1)^n\sin(n-m)\phi\sin 2\eta] \tag{4.11}$$

Consider the first term in the bracket [ ]. Since $\cos\phi$ is an even function of λ, $\cos(n-m)\phi$ is even and $\cos 2\eta$ is odd (see Eq. (2.11)). Thus the integrand is odd and gives zero upon integrating over $(-b,-a)\cup(a,b)$. Similarly the second term in the [ ] is also odd, because $\sin\phi$ is odd which implies $\sin(n-m)\phi$ is odd and $\sin 2\eta$ is even from Eq. (2.11). Thus $(\psi_n,\psi_m)=0$ for $n-m$ odd.

This completes the proof of the orthonormality of $\psi_n(\lambda)$.

5 Kernel

We will obtain an expression for $K(\mu,\nu)$ analogous to the Christoffel-Darboux formula from Eq. (2.5), which is more convenient for the present two cut structure. Multiplying and dividing Eq. (2.5) by $\mu^2-\nu^2$, the kernel for $N=2P$ even is

$$\begin{aligned} K(\mu,\nu) &= \frac{1}{2P}\Big[\sum_{n=0}^{(P-1)} \frac{\mu^2\psi_{2n}(\mu)\psi_{2n}(\nu)-\psi_{2n}(\mu)\nu^2\psi_{2n}(\nu)}{(\mu^2-\nu^2)} \\ &+ \sum_{n=0}^{(P-1)} \frac{\mu^2\psi_{2n+1}(\mu)\psi_{2n+1}(\nu)-\psi_{2n+1}(\mu)\nu^2\psi_{2n+1}(\nu)}{(\mu^2-\nu^2)}\Big] \end{aligned} \tag{5.1}$$

Using the recurrence relationship for the even and odd orthogonal polynomials and with some algebra we get

$$\begin{aligned} K(\mu,\nu) = \frac{1}{2P}\frac{1}{(\mu^2-\nu^2)}&[\sqrt{A_{2P}B_{2P+1}}(\psi_{2P+1}(\mu)\psi_{2P-1}(\nu) \\ - \quad &\psi_{2P-1}(\mu)\psi_{2P+1}(\nu)) + \sqrt{B_{2P-1}A_{2P}}(\psi_{2P}(\mu)\psi_{2P-2}(\nu) \\ - \quad &\psi_{2P-2}(\mu)\psi_{2P}(\nu))] \end{aligned} \tag{5.2}$$

Following a similar procedure for $N=2P+1$ odd the kernel $K(\mu,\nu)$ is

$$\begin{aligned} K(\mu,\nu) &= \frac{1}{(2P+1)}\frac{1}{(\mu^2-\nu^2)}[\sqrt{A_{2P}B_{2P+1}}(\psi_{2P+1}(\mu)\psi_{2P-1}(\nu) \\ &- \psi_{2P-1}(\mu)\psi_{2P+1}(\nu)) + \sqrt{B_{2P+1}A_{2P}}(\psi_{2P+2}(\mu)\psi_{2P}(\nu) \\ &- \psi_{2P}(\mu)\psi_{2P+2}(\nu))] \end{aligned} \tag{5.3}$$

The kernel in this form is more useful than the standard form for the symmetric two cut hermitian matrix model due to the explicit form of f and ϕ (Eqs. (2.8) and (2.10)) in which 2ϕ appears rather than ϕ.

A convenient form for the kernel with two cuts as derived above is (for N even)

$$
\begin{aligned}
& K(\mu,\nu) = K^{(o)}(\mu,\nu) + K^{(e)}(\mu,\nu) \\
= \; & \frac{1}{N}\frac{1}{(\mu^2-\nu^2)}[\sqrt{A_N B_{N+1}}(\psi_{N+1}(\mu)\psi_{N-1}(\nu) - \psi_{N-1}(\mu)\psi_{N+1}(\nu))] \\
+ \; & \frac{1}{N}\frac{1}{(\mu^2-\nu^2)}[\sqrt{A_N B_{N-1}}(\psi_{N}(\mu)\psi_{N-2}(\nu) - \psi_{N-2}(\mu)\psi_{N}(\nu))]. \quad (5.4)
\end{aligned}
$$

Using the asymptotic ansatz and doing some simple trigonometry with $\psi_{N+1}(\mu) = \frac{1}{\sqrt{f(\mu)}}\cos(N\zeta+\phi+\chi-\eta)(\mu)$, $\psi_{N-1}(\nu) = \frac{1}{\sqrt{f(\nu)}}\cos(N\zeta-\phi+\chi-\eta)(\nu)$, $\psi_N(\mu) = \frac{1}{\sqrt{f(\mu)}}\cos(N\zeta+\chi+\eta)(\mu)$ and $\psi_{N-2}(\nu) = \frac{1}{\sqrt{f(\nu)}}\cos(N\zeta-2\phi+\chi+\eta)(\nu)$ we get the following

$$
\begin{aligned}
K^{(o)}(\mu,\nu) \; = \; & \frac{-1}{2N}\frac{\sqrt{A_N B_{N+1}}}{(\mu^2-\nu^2)\sqrt{f(\mu)f(\nu)}} \\
[\; & (\cos 2\phi(\mu) - \cos 2\phi(\nu))[\cos N(h^{(o)}(\mu) + h^{(o)}(\nu)) \\
+ \; & \cos N(h^{(o)}(\mu) - h^{(o)}(\nu))] \\
+ \; & (\sin 2\phi(\mu) - \sin 2\phi(\nu)) \sin N(h^{(o)}(\mu) + h^{(o)}(\nu)) \\
+ \; & (\sin 2\phi(\mu) + \sin 2\phi(\nu)) \sin N(h^{(o)}(\mu) - h^{(o)}(\nu))] \quad (5.5)
\end{aligned}
$$

where $Nh^{(o)}(\mu) = (N\zeta + \chi + \phi - \eta)(\mu)$ and

$$
\begin{aligned}
K^{(e)}(\mu,\nu) \; = \; & \frac{-1}{2N}\frac{\sqrt{A_N B_{N-1}}}{(\mu^2-\nu^2)\sqrt{f(\mu)f(\nu)}} \\
[\; & (\cos 2\phi(\mu) - \cos 2\phi(\nu))[\cos N(h^{(e)}(\mu) + h^{(e)}(\nu)) \\
+ \; & \cos N(h^{(e)}(\mu) - h^{(e)}(\nu))] \\
+ \; & (\sin 2\phi(\mu) - \sin 2\phi(\nu)) \sin N(h^{(e)}(\mu) + h^{(e)}(\nu)) \\
+ \; & (\sin 2\phi(\mu) + \sin 2\phi(\nu)) \sin N(h^{(e)}(\mu) - h^{(e)}(\nu))] \quad (5.6)
\end{aligned}
$$

where $Nh^{(e)}(\mu) = (N\zeta + \chi + \eta)(\mu)$. Using the fact that $\lim_{\nu\to\mu} K(\mu,\nu) = \rho(\mu)$ one finds that $\zeta'(\mu) = -\pi\rho(\mu)$. In more detail, the first term in $K^{(o)}(\mu,\nu)$ and $K^{(e)}(\mu,\nu)$ is of order $O(\frac{1}{N})$ while the second and third terms give a term of order $O(1)$ which arises due to a Taylor expansion about $\nu = \mu + \delta$. The $O(1)$ terms each give a factor $\frac{-1}{2\pi}h'^{(e)}(\mu)$ and $\frac{-1}{2\pi}h'^{(o)}(\mu)$ on taking the $\nu \to \mu$ limit. These terms then combine to give $\frac{-1}{\pi}\zeta'(\mu)$. This determines the unknown function $\zeta(\mu)$ in the asymptotic ansatz $\psi_n(\mu)$ in terms of the density of eigenvalues, and yields Eq. (2.9).

Eqs. (5.5) and (5.6) together with $K = K^{(o)} + K^{(e)}$, and Eqs. (2.8-2.11) describe the full kernel function for the hermitian one matrix model with eigenvalues distributed in two disjoint cuts. The result is valid for any polynomial potential which can support two symmetric cuts. When $a = 0$ the two cuts in the density of eigenvalues merge into one, there is a phase transition from a double band to a single band. The two-cut ansatz, Eq. (2.7) for $\psi_n(\lambda)$, changes to the one-cut ansatz of ref. [1] Eq. (2.6) at $a = 0$. Thus the exact kernel derived above contains all the information of phase transitions and universality for both the fine and coarse grained correlators.

6 Correlation Functions

The full two-point connected correlation function for the two-cut hermitian matrix model is

$$\begin{aligned} \rho_c(\mu,\nu) &= -[K(\mu,\nu)]^2 = -([K^{(o)}(\mu,\nu)]^2 + [K^{(e)}(\mu,\nu)]^2 \\ &+ 2K^{(o)}(\mu,\nu)K^{(e)}(\mu,\nu)) \end{aligned} \tag{6.1}$$

with $K^{(o)}(\mu,\nu)$ and $K^{(e)}(\mu,\nu)$ given by Eq. (5.5) and Eq. (5.6) respectively.

We now discuss the correlator in various limits.

(a). First consider the fine grained correlator i.e. where the separation $\mu-\nu$ of the order of the eigenvalue spacing $\approx O(\frac{1}{N})$;

$$(\mu-\nu) \approx O(\frac{1}{N}) \equiv 2\delta \tag{6.2}$$

Then only the third terms in $K^{(o)}(\mu,\nu)$ and $K^{(e)}(\mu,\nu)$ contribute to $K(\mu,\nu)$. Thus (for N even)

$$K(\mu,\nu) \to \frac{\sin[2N\pi\delta\rho(\bar{\mu})]}{2N\pi\delta} \tag{6.3}$$

where $\bar{\mu} = \frac{\mu+\nu}{2}$. Over the range δ, $\rho(\bar{\mu})$ is just constant which can be scaled away. This extends the validity of Dyson's short distance universality to eigenvalues distributed in two disconnected cuts. The two point correlation function is

$$\rho(\mu,\nu) = \rho(\mu)\rho(\nu)[1 - (\frac{\sin x}{x})^2] \tag{6.4}$$

with $x = 2N\pi\delta\rho(\bar{\mu})$.

(b). For $\delta << O(\frac{1}{N})$ and $x << 1$ one gets

$$\rho(\mu,\nu) \approx \frac{1}{3}\pi^2\rho^4(\bar{\mu})[N(\mu-\nu)]^2. \tag{6.5}$$

It is evident that the exact short distance correlator has no singularity as $\nu \to \mu$.
(c). In the smoothing regime, i.e., for $\mu - \nu >> O(\frac{1}{N})$ we average over μ, ν in an interval Δ such that $\frac{1}{N} << \Delta << 1$. Replacing $< \cos^2 Nh >$ and $< \sin^2 Nh >$ by a half and $< \cos Nh >=< \sin Nh >= 0$

$$\begin{aligned} \rho_c^{smooth}(\mu,\nu) &= -< [K(\mu,\nu)]^2 > \\ &= -\frac{1}{2N^2\pi^2}\frac{\mu\nu}{(\mu^2-\nu^2)^2}\frac{1}{\sin 2\phi(\mu)\sin 2\phi(\nu)} \\ &[\quad (1-\cos 2\phi(\mu)\cos 2\phi(\nu))(1+\cos\bar{\phi}(\mu)\cos\bar{\phi}(\nu)) \\ &+\quad \sin 2\phi(\mu)\sin 2\phi(\nu)\sin\bar{\phi}(\mu)\sin\bar{\phi}(\nu)], \end{aligned} \tag{6.6}$$

where $\bar{\phi}(\mu) = \phi(\mu) - 2\eta(\mu)$, is the global smoothed correlator valid all over the cuts. It is interesting to note that even when μ and ν belong to the same segment the smoothed correlator has changed from the single band universal result (ref. [1] Eq. (2.22)).

This proves the universality of smoothed global two point correlators (as the correlator depends only on the end points of the cuts $(-b,-a)$ and (a,b)) for symmetric two-cut matrix models, generalizing the universality for the one cut matrix models proven in [19, 1]. Note that the function χ undetermined in our approach is absent in the universal limits $(a),(b)$ and (c).

7 Conclusion

In this work, the orthogonal polynomial method for large N hermitian matrix models with arbitrary potentials that support eigenvalues distributed in two symmetric cuts has been developed. An asymptotic form for the orthogonal polynomials $\psi_n(\lambda)$ valid for n close to N and λ inside the cuts has been obtained. This is done by making an ansatz in terms of unknown functions f, ϕ, η, χ and ζ. The first three are then constrained using the recursion relations and orthonormality, which determine them to be universal functions of λ independent of the form of the potential except for the endpoints of the cuts. χ, though not determined in this approach is also conjectured to be universal. The crucial difference between the one-cut case and the two-cut case appears in the behaviour of the recursion coefficients, which have a "period two" character, i.e., are described by two continuous functions in the large N limit instead of one. This modifies the functional

form of the functions f and ϕ and introduces the new nontrivial function η in the asymptotic ansatz that vanishes in the single cut case.

The orthogonal polynomials are then used in a Christoffel-Darboux formula (a version of this formula especially suited for two symmetric cuts is used) to determine the eigenvalue kernel $K(\mu, \nu)$. The relation between this kernel and the eigenvalue density fixes the last unknown function ζ in the ansatz. This is the only non universal function in the asymptotic formula for the orthogonal polynomials and hence also in the kernel and the correlation functions.

The kernel determines global fine grained eigenvalue n-point correlation functions through well known formulas. The two point function is discussed in detail. It is shown to be universal in two limits, the local fine grained limit and the global smoothed limit (in the latter it depends on the potential only through the endpoints of the cuts). It is hoped that the above can be generalized to multi-cut as well as complex matrix models.

<u>Note added</u>:

After this work was completed I became aware of [20], where the orthogonal polynomials are obtained for the case of quartic potential $V(M) = \frac{g_2}{2}M^2 + \frac{g_4}{4}M^4$ with two cuts by different methods. I thank Prof. E. Brezin for information about this work. The method and results of the present paper are valid more generally, i.e., for arbitrary Z_2 symmetric polynomial potentials.

I would like to thank the referee for pointing out the following references: i) [22] for a renormalization group approach to universality of smoothed correlators in multi-cut models, ii) [12] for applications of two-cut models to QCD, and iii) [21] for computation of smoothed Green function for multi-cut models. It is not clear that the connected density-density correlator derived from [21] is the same as the one presented here. This may be due to the different limiting procedures used (e.g. the $N \to \infty$ and $Asym \to 0$ limits do not commute [11]). I would also like to thank G. Akemann for an email discussion on this point.

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References

[1] E. Brezin and A. Zee, *Nucl. Phys.* **B402** (1993) 613

[2] B. D. Simons, P. A. Lee and B. L. Altschuler, *Phys. Rev. Lett.* **70** (1993) 4122; *Nucl. Phys.* **B409** (1993) 487

[3] O. Narayan and B. S. Shastry, *Phys. Rev. Lett.* **71** (1993) 2106

[4] C. W. J. Beenakker, *Nucl. Phys.* **B422** (1994) 515;*Phys. Rev. Lett.* **70** (1993) 1155

[5] E. Brezin and A. Zee, *Nucl. Phys.* **B424** (1994) 435;*Phys. Rev.* **E49** (1994) 2588

[6] A. Pandey, *Chaos, Solitons and Fractals* **5**, 1275, 1995

[7] N. Deo, S. Jain and B. S. Shastry, *Phys. Rev.* **E52** (1995) 4836

[8] N. Deo, *Nucl. Phys.* **B464** (1996) 463

[9] S. Jain, *Mod. Phys. Lett.* **A11** (1996) 1201

[10] E. Brezin, J. D'Anna and A. Zee, *Nucl. Phys.* **B443** (1995) 433

[11] R. C. Brower, N. Deo, S. Jain and C. I. Tan, *Nucl. Phys.* **B405** (1993) 166

[12] A. Jackson and J. Verbaarschot, *Phys. Rev.* **D53** (1996) 7223 and A. Jackson, M. Sener and J. Verbaarschot, *Nucl. Phys.* **B479** (1996) 707

[13] J. Jurkiewicz, M. A. Nowak and I. Zahed, hep-th/9603308

[14] L. F. Cugliandolo, J. Kurchan, G. Parisi and F. Ritort, *Phys. Rev. Lett.* **74** (1995) 1012

[15] M. R. Douglas, N. Seiberg and S. H. Shenker, *Phys. Lett.* **B244** (1990) 381

[16] K. Demeterfi, N. Deo, S. Jain and C. I. Tan, *Phys. Rev.* **D42** (1990) 4105

[17] B. Eynard, Saclay preprint, T93/145, (hep-th/9401165)

[18] G. M. Cicuta, L. Molinari and E. Montaldi, *Mod. Phys. Lett.* **A1** (1986) 125

[19] J. Ambjorn, J. Jurkiewicz and Yu. M. Makeenko, *Phys. Lett.* **B251** (1990) 517

[20] Bleher and Its, Math IUPUI preprint, 1997

[21] J. Ambjorn and G. Akemann, *J. Phys.* **A29** (1996) L555

[22] S. Higuchi, C. Itoi, S. Nishigaki, N. Sakai, *Phys. Lett.* **B398** (1997) 123.